Matrix Factorization.

We have a theorem that says a matrix is invertible if and only if it can be factored as a product of elementary matrices. In this supplement we will see an example of how to factor an invertible matrix into elementary matrices. The basic idea is to row-reduce to the identity, keep track of the row moves and their order, Since each row move corresponds to multiplication by an elementary matrix, writing the elementary matrices corresponding to the row moves tell us how to factor the matrix.

Example: Write the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ as a product of elementary matrices.

First, we row-reduce, noting the moves and their order:

$$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \overset{r_1 \leftrightarrow r_2}{\rightarrow} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \overset{r_1 - r_2 \mapsto r_1}{\rightarrow} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \overset{r_2 - 2r_1 \mapsto r_2}{\rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, since doing each move is equivalent to multiplying on the left by the corresponding elementary matrix, this says

$$E_{r_2 - 2r_1 \mapsto r_2} E_{r_1 - r_2 \mapsto r_1} E_{r_1 \leftrightarrow r_2} A = I.$$

We can solve the equation for A by multiplying on the left by the inverses of the elementary matrices:

$$\begin{aligned} E_{r_2 - 2r_1 \mapsto r_2} E_{r_1 - r_2 \mapsto r_1} E_{r_1 \leftrightarrow r_2} A &= I \\ E_{r_1 - r_2 \mapsto r_1} E_{r_1 \leftrightarrow r_2} A &= E^{-1}_{r_2 - 2r_1 \mapsto r_2} \\ E_{r_1 \leftrightarrow r_2} A &= E^{-1}_{r_1 - r_2 \mapsto r_1} E^{-1}_{r_2 - 2r_1 \mapsto r_2} \\ A &= E^{-1}_{r_1 \leftrightarrow r_2} E^{-1}_{r_1 - r_2 \mapsto r_1} E^{-1}_{r_2 - 2r_1 \mapsto r_2}. \end{aligned}$$

Finally, note that the inverse of an elementary matrix is the elementary matrix of the opposite row operation, replacing addition with subtraction and multiplication with division.

$$\begin{aligned} A &= E^{-1}_{r_1 \leftrightarrow r_2} E^{-1}_{r_1 - r_2 \mapsto r_1} E^{-1}_{r_2 - 2r_1 \mapsto r_2} \\ A &= E_{r_1 \leftrightarrow r_2} E_{r_1 + r_2 \mapsto r_1} E_{r_2 + 2r_1 \mapsto r_2}. \end{aligned}$$

Thus, we should have

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Let us verify that this works:

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

as required.