1. Suppose S and T are subspaces of an $\mathbb{F}$-vector space V. Is their union $S \cup T$ a subspace? Explain.

In general, $S \cup T$ is not a subspace. For example, consider $S = \mathrm{Span}((1,0))$ and $T = \mathrm{Span}((0,1))$ in $\mathbb{R}^2$; then $S \cup T$ is the set of all points on either the x- or y-axes, but $(1,0)+(0,1) = (1,1) \notin S \cup T$.

2. Using induction, prove that $3^n - 1$ is divisible by 2.

Many proofs are possible. One goes like this:

<u>Case $n = 1$:</u> If $n = 1$, the $3^n - 1 = 3^1 - 1 = 3 - 1 = 2$ and 2 is divisble by 2.

<u>Induction Step:</u> Suppose $3^n - 1 = 2k$ and consider $3^{n+1} - 1$. We can write

$$3^{n+1} - 1 = 3(3^n) - 1 = 3(3^n - 1) + 3 - 1 = 3(2k) + 2 = 2(3k+1)$$

and hence $3^n - 1$ divisible by 2 implies $3^{n+1} - 1$ divisible by 2, as required.

3. Let $S = \mathrm{Span}((1,0,1,1),(1,2,0,1)) \subset \mathbb{R}^4$ and $T = \mathrm{Span}((1,1,0,1),(1,1,1,1))) \subset \mathbb{R}^4$. Compute the dimension of $S \cap T$.

<u>$\dim(S)$:</u>

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \Rightarrow \dim(S) = 2$$

<u>$\dim(T)$:</u>

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \dim(T) = 2$$

<u>$\dim(S+T)$:</u>

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\dim(S \cap T) = \dim(S) + \dim(T) - \dim(S+T) = 2 + 2 - 3 = 1$.

4. Determine whether the set $\{(1,0,2),(2,0,1),(0,2,1)\} \subset \mathbb{Q}^3$ is linearly dependent or independent.

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -3 \\ 2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -3 \\ 0 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & -3 \end{bmatrix}$$

Since we have echelon form with all three rows nonzero, the span is 3-dimensional and hence the original vectors are linearly independent.

5. Let $M = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \in M_3(\mathbb{Z}_3)$, i.e. the entries are in $\mathbb{Z}_3$. Determine the rank of M.

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the rank of M is 2.

6. Find the solution set of the system of linear equations over $\mathbb{R}$ given by the augmented matrix

$$\left[\begin{array}{rrrr|r} 2 & 4 & 2 & -2 & 0 \\ 1 & -2 & 1 & 0 & -1 \\ 3 & -2 & 3 & -1 & -2 \end{array}\right].$$

$$\left[\begin{array}{rrrr|r} 2 & 4 & 2 & -2 & 0 \\ 1 & -2 & 1 & 0 & -1 \\ 3 & -2 & 3 & -1 & -2 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 0 & -1 \\ 2 & 4 & 2 & -2 & 0 \\ 3 & -2 & 3 & -1 & -2 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 0 & -1 \\ 0 & 8 & 0 & -2 & 2 \\ 3 & -2 & 3 & -1 & -2 \end{array}\right] \to$$

$$\to \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 0 & -1 \\ 0 & 8 & 0 & -2 & 2 \\ 0 & 4 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 0 & -1 \\ 0 & 4 & 0 & -1 & 1 \\ 0 & 4 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 0 & -1 \\ 0 & 4 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to$$

$$\to \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the homogeneous solutions have the form

$$\left(-\alpha + \frac{1}{2}\beta, \frac{1}{4}\beta, \alpha, \beta\right) = \alpha\,(-1, 0, 1, 0) + \beta\left(\frac{1}{2}, \frac{1}{4}, 0, 1\right),$$

while the particular solution is $(-\frac{1}{2}, \frac{1}{4}, 0, 0)$ and the solution set is

$$\left(-\frac{1}{2}, \frac{1}{4}, 0, 0\right) + \operatorname{Span}\left((-1, 0, 1, 0), \left(\frac{1}{2}, \frac{1}{4}, 0, 1\right)\right).$$

7. Using the vector space axioms, prove that if V is an $\mathbb{F}$-vector space, then $0\vec{u} = \vec{0}$ for all $\vec{u} \in V$.

Suppose V is an $\mathbb{F}$-vector space and let $\vec{u} \in V$. Then on the one hand, we have $1\vec{u} = \vec{u}$, so

$$\vec{u} = 1\vec{u} = (0+1)\vec{u} = 0\vec{u} + 1\vec{u} = 0\vec{u} + \vec{u}$$

while on the other hand, we have from the vector space axioms

$$\vec{u} = \vec{0} + \vec{u}.$$

Then

$$\begin{aligned} 0\vec{u} + \vec{u} &= \vec{0} + \vec{u} \\ (0\vec{u} + \vec{u}) + -\vec{u} &= (\vec{0} + \vec{u}) + -\vec{u} \\ 0\vec{u} + (\vec{u} + -\vec{u}) &= \vec{0} + (\vec{u} + -\vec{u}) \\ 0\vec{u} + \vec{0} &= \vec{0} + \vec{0} \\ 0\vec{u} &= \vec{0}. \end{aligned}$$

8. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (x + y - z, 2x - y + z)$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $g(u, v) = (u - 3v, 2v, 2u + v)$. Find the matrix of $g \circ f : \mathbb{R}^3 \to \mathbb{R}^3$.

The matrices of f and g respectively are

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$$

so the matrix of $g \circ f$ is

$$\begin{aligned} BA &= \left[\begin{array}{cc} 1 & -3 \\ \hline 0 & 2 \\ \hline 2 & 1 \end{array}\right] \left[\begin{array}{c|c|c} 1 & 1 & -1 \\ 2 & -1 & 1 \end{array}\right] \\ &= \begin{bmatrix} 1(1) - 3(2) & 1(1) - 3(-1) & 1(-1) - 3(1) \\ 0(1) + 2(2) & 0(1) + 2(-1) & 0(-1) + 2(1) \\ 2(1) + 1(2) & 2(1) + 1(-1) & 2(-1) + 1(1) \end{bmatrix} \\ &= \begin{bmatrix} -5 & 4 & -4 \\ 4 & -2 & 2 \\ 4 & 1 & -1 \end{bmatrix} \end{aligned}$$

Bonus [0-2 points each]:

A. Let $X = \{0, 1, t, 1 + t\}$ with $\mathbb{Z}_2$ coefficients and $t^2 = 1 + t$. Does X satisfy the multiplicative invertibility requirements for being a field?

It does. We need every nonzero element to have a multiplicative inverse. We have $1(1) = 1$ so $1^{-1} = 1$, and $t(1 + t) = t^2 + t = (1 + t) + t = 1 + 2t = 1$, so $t^{-1} = 1 + t$ and $(1 + t)^{-1} = t$.

B. Let $\mathcal{C}^\infty(\mathbb{R})$ be the vector space of infinitely-often differentiable functions of one real variable with $(f + g)(x) = f(x) + g(x)$ and $f(\lambda x) = \lambda f(x)$. Let $\frac{d}{dx} : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ be the derivative, i.e. $\frac{d}{dx}(f) = f'$. Is $\frac{d}{dx}$ a linear transformation? Explain.

It is. As we know from calculus,

$$\frac{d}{dx}\left(f(x) + g(x)\right) \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$$

and

$$\frac{d}{dx}\left(\alpha f(x)\right) = \alpha \frac{d}{dx}\left(f(x)\right).$$