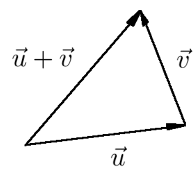
Last time we saw some examples of inner products and proved the *Cauchy-Schwartz Inequality.* Another useful fact about inner product spaces is the *triangle inequality.* Geometrically, this says the length of any one side of a triangle is less than or equal to the sum of the other two sides.



In vector terms, this says:

Theorem: (Triangle Inequality) Let $\vec{u}, \vec{v}$ be vectors in an inner product space. Then

$$||\vec{u}+\vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||.$$

Proof: First, note that for real numbers $\alpha_1, \ldots, \alpha_n$ we have

$$|\alpha_1 + \ldots + \alpha_n| \leq |\alpha_1| + \ldots + |\alpha_n|$$

with equality if all α_i have the same sign, while if some α_is have different signs, subtraction makes the left side smaller in absolute value.

Now, consider $\vec{u}+\vec{v}$. By definition we have

$$\begin{aligned} ||\vec{u}+\vec{v}||^2 &= \langle \vec{u}+\vec{v}, \vec{u}+\vec{v}\rangle \\ &= |\langle \vec{u}+\vec{v}, \vec{u}+\vec{v}\rangle| \\ &= |\langle \vec{u}, \vec{u}\rangle + 2\langle \vec{u}, \vec{v}\rangle + \langle \vec{v}, \vec{v}\rangle| \\ &\leq |\langle \vec{u}, \vec{u}\rangle| + 2|\langle \vec{u}, \vec{v}\rangle| + |\langle \vec{v}, \vec{v}\rangle| \\ &\leq ||\vec{u}||^2 + 2||\vec{u}||\ ||\vec{v}|| + ||\vec{v}||^2 \\ &= (||u|| + ||v||)^2 \end{aligned}$$

Thus we have

$$||\vec{u}+\vec{v}||^2 \leq (||u|| + ||v||)^2$$

which says

$$0 \leq (||u|| + ||v||)^2 - ||\vec{u}+\vec{v}||^2.$$

We can the factor the right hand side to get

$$(||u|| + ||v|| + ||\vec{u}+\vec{v}||)(||u|| + ||v|| - ||\vec{u}+\vec{v}||) \geq 0$$

so both factors must have the same sign. If both are negative, then

$$||u|| + ||v|| + ||\vec{u}+\vec{v}|| \leq 0$$

and since each length is greater than or equal to zero, all three must be zero, and the statement holds; if both are positive, then

$$||u|| + ||v|| - ||\vec{u}+\vec{v}|| \geq 0$$

and then

$$||u|| + ||v|| \geq ||\vec{u}+\vec{v}||$$

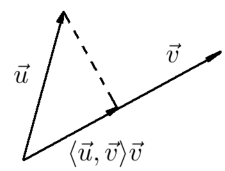
as required. □

Orthonormal Bases and Orthogonal Transformations.

Two vectors $\vec{u}, \vec{v}$ in an inner product space are *orthogonal* if their inner product is zero, i.e. if $\langle \vec{u}, \vec{v}\rangle = 0$. Since $\cos^{-1}(0) = \frac{\pi}{2}$, orthogonal vectors are at right angles. A basis $B = (\vec{b}_1, \ldots, \vec{b}_n)$ for an inner product space V is *orthonormal* if

- $||\vec{b}_i|| = 1$ for all i, i.e. every basis vector is a unit vector, and
- $\langle \vec{b}_i, \vec{b}_j \rangle = 0$ for all i, j, i.e every pair of basis vectors is orthogonal.

Example: The standard basis of $\mathbb{R}^n$ is an orthonormal basis with respect to the dot product. An inner product can be understood in terms of *projections*. Specifically, if $\vec{u}, \vec{v}$ are vectors in an inner product space, then $\langle \vec{u}, \vec{v} \rangle \vec{v}$ is an orthogonal projection of $\vec{u}$ onto the subspace spanned by $\vec{v}$:



Given any basis $B = (\vec{b}_1, \ldots, \vec{b}_n)$, we can build an orthonormal basis B' by the *Gramm-Schmidt Process*:

- Start by setting $\vec{b}_1' = ||\vec{b}_1||^{-1} \vec{b}_1$.
- For each old basis vector, subtract off the projections onto the previous new basis vectors and then normalize, i.e. multiply by the inverse of the norm:

$$u_2 = \vec{b_2} - \langle \vec{b}_2, \vec{b}_1' \rangle \vec{b_1}', \quad \vec{b}_2' = ||\vec{u}_2||^{-1} \vec{u}_2,$$

and in general

$$u_k = \vec{b_2} - \sum_{j=1}^{k-1} \langle \vec{b}_k, \vec{b}_j' \rangle \vec{b_j}', \quad \vec{b}_k' = ||\vec{u}_k||^{-1} \vec{u}_k.$$

Finally, we have

Definition: A linear transformation $f : V \to W$ between inner product spaces V and W is an *orthogonal transformation* if it preserves inner products, that is, is

$$\langle f(\vec{u}), f(\vec{v}) \rangle_W = \langle \vec{u}, \vec{v} \rangle_V$$

for all $\vec{u}, \vec{v} \in V$, where $\langle , \rangle_V$ and $\langle , \rangle_W$ are the inner products on V and W respectively.

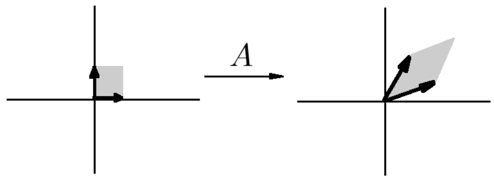
Orthogonal transformations preserve length and angles, i.e.,

$$||f(\vec{u})|| = ||\vec{u}|| \quad \text{and} \quad \theta(f(\vec{u}), f(\vec{v})) = \theta(\vec{u}, \vec{v})$$

if f is an orthogonal transformation. Further, f sends orthogonal sets to orthogonal sets. The matrix of an orthogonal transformation $f : V \to V$ with respect to an orthonormal basis has a very nice property: $A^T A = I$. That is, the inverse of the matrix of an othrogonal transformation with respect to an orthonormal basis is just the transpose of the matrix, $A^{-1} = A^T$.

Determinants.

Let $A = M_n(\mathbb{F})$ be a matrix. The columns of A represent the images $A\vec{e}_1, A\vec{e}_2, \ldots, A\vec{e}_n$ of the standard basis vectors under the linear transformation given by left-multiplication by A. We can get an intutitive picture of what A does to $\mathbb{F}^n$ by looking at what A does to the standard basis. For example, the effect of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ on the standard basis of $\mathbb{R}^2$ looks like:



A set of basis vectors can be completed to form a parallelogram (or in dimensions greater than 2, a "squashed box" known as a *parallelopiped*). We can compare the signed areas (or signed n-volumes) of the parallelopipeds to get a numerical measure of the effect of the linear transformation A. The ratio of these volumes, which is actually just the volume of the parallelopiped determined by the columns of A since the standard basis box has volume 1, is called the *determinant* of A, denoted $|A|$ or $\det(A)$ or sometimes $D(A)$.