1. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x, y, z) = (x + y - z, 2x + y, x - y + 3z)$. Find a polynomial p such that $p(f)(1, 1, 1) = (0, 0, 0)$.

For this, we observe that $f(1, 1, 1) = (1, 3, 3)$, $f^2(1, 1, 1) = f(1, 3, 3) = (1, 5, 7)$ and $f^3(1, 1, 1) = f(1, 5, 7) = (-1, 7, 17)$ Then we have an equation of vectors

$$\alpha_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 7 \\ 15 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which we can solve by row-reduction:

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 3 & 5 & 7 \\ 1 & 3 & 7 & 17 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 4 & 8 \\ 1 & 3 & 7 & 17 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 4 & 8 \\ 0 & 2 & 6 & 18 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 2 & 10 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -6 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to$$

so we have $\alpha_0 = 0, \alpha_1 = 6$, $\alpha_2 = -5$ and $\alpha_3 = 1$ and $p(f) = 6f - 5f^2 + f^3$.

2. Find the eigenvalues and eigenspaces of the matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

From the characteristic polynomial $D(A - \lambda I) = (1 - \lambda)^2(2 - \lambda)$, we see that there are two eigenvalues, $\lambda = 1$ and $\lambda = 2$.

<u>$\lambda = 1$</u>:

$$\begin{bmatrix} 1-1 & -1 & -1 \\ 0 & 2-1 & 1 \\ 0 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the $\lambda = 1$ eigenspace is $\text{Span}((0, -1, 1))$.

<u>$\lambda = 2$</u>:

$$\begin{bmatrix} 1-2 & -1 & -1 \\ 0 & 2-2 & 1 \\ 0 & 0 & 1-2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the $\lambda = 2$ eigenspace is $\text{Span}((-1, 1, 0))$.

3. Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

First, we need to find the eigenvalues:

$$D(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ -1 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2 - \lambda) \begin{vmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} = (2 - \lambda)^2(1 - \lambda).$$

<u>$\lambda = 1$</u>:

$$\begin{bmatrix} 1-1 & 0 & 1 \\ -1 & 2-1 & 1 \\ 0 & 0 & 2-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the $\lambda = 1$ eigenspace is Span((1, 1, 0)).
$\underline{\lambda = 2}$:

$$\begin{bmatrix} 1-2 & 0 & 1 \\ -1 & 2-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the $\lambda = 2$ eigenspace is Span((0, 1, 0), (1, 0, 1)). Then we have

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Prove that eigenvectors corresponding to different eigenvalues are linearly independent.

Let $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n$ be a list of distinct eigenvalues with eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ and suppose that

$$\alpha_1\vec{v}_1 + \ldots + \alpha_n\vec{v}_n = \vec{0}.$$

We must show that $\alpha_k = 0$ for $k = 1, \ldots, n$. Let us proceed by induction on n.

If $n = 1$, then $\alpha_1\vec{v}_1 = \vec{0}$ and the fact that as an eigenvector, $\vec{v}_1 \neq \vec{0}$ imply that $\alpha_1 = 0$, so the statement is true in the $n = 1$ case.

Now, suppose the statement holds for sets of fewer than n eigenvectors and consider $f(\alpha_1\vec{v}_1+\ldots+\alpha_n\vec{v}_n)$.

$$\vec{0} = f(\alpha_1\vec{v}_1 + \ldots + \alpha_n\vec{v}_n) = f(\alpha_1\vec{v}_1) + \ldots + f(\alpha_n\vec{v}_n) = \alpha_1 f(\vec{v}_1) + \ldots + \alpha_n f(\vec{v}_n) = \alpha_1\lambda_1\vec{v}_1 + \ldots + \alpha_n\lambda_n\vec{v}_n.$$

Either all of the eigenvalues are nonzero or one of the eigenvalues, say λ_n, is zero. If $\lambda_n = 0$ then our induction hypothesis says $\alpha_1, \ldots, \alpha_{n-1}$ must all be zero; then

$$\alpha_1\vec{v}_1 + \ldots + \alpha_n\vec{v}_n = \vec{0}$$

implies $\alpha_n\vec{v}_n = 0$, and the fact that $\vec{v}_n \neq \vec{0}$ implies that $\alpha_n = 0$.

If all of the eigenvalues λ_k are nonzero, then suppose some coefficient, say α_n, is nonzero; we will derive a contradiction. Then we have

$$-\frac{\alpha_1}{\alpha_n}\vec{v}_1 + \ldots - \frac{\alpha_{n-1}}{\alpha_n}\vec{v}_{n-1} = \vec{v}_n$$

and then

$$\begin{aligned}
\vec{0} &= f(\vec{v}_n) - \lambda v_n \\
&= f\left(-\frac{\alpha_1}{\alpha_n}\vec{v}_1 + \ldots - \frac{\alpha_{n-1}}{\alpha_n}\vec{v}_{n-1}\right) - \lambda_n\left(-\frac{\alpha_1}{\alpha_n}\vec{v}_1 + \ldots - \frac{\alpha_{n-1}}{\alpha_n}\vec{v}_{n-1}\right) \\
&= \left(-\frac{\alpha_1}{\alpha_n}\lambda_1\vec{v}_1 + \ldots - \frac{\alpha_{n-1}}{\alpha_n}\lambda_{n-1}\vec{v}_{n-1}\right) - \lambda_n\left(-\frac{\alpha_1}{\alpha_n}\vec{v}_1 + \ldots - \frac{\alpha_{n-1}}{\alpha_n}\vec{v}_{n-1}\right) \\
&= \alpha_1\frac{-\lambda_1 + \lambda_n}{\alpha_n}\vec{v}_1 + \ldots + \alpha_{n-1}\frac{-\lambda_{n-1} + \lambda_n}{\alpha_n}\vec{v}_{n-1}.
\end{aligned}$$

Since $\lambda_j \neq \lambda_n$ for $j = 1, \ldots, n-1$, we have $\alpha_1 = \ldots - \alpha_{n-1} = 0$ by our induction hypothesis. But then we have $0 + \ldots + 0 + \alpha_n\vec{v}_n = \vec{0}$ with $\alpha_n \neq 0$, contradicting the fact that $\vec{v}_n$ is an eigenvector and thus nonzero. Thus, no α_j can be nonzero, and the set is linearly independent.