Math 60 **Homework 8 Solutions**

1. Compute the determinant of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$ using the complete expansion.

$$\begin{aligned}
\begin{vmatrix} 2&0&1\\-1&1&0\\2&1&3\end{vmatrix} &= 2\begin{vmatrix}1&0&0\\-1&1&0\\2&1&3\end{vmatrix} + 0\begin{vmatrix}0&1&0\\-1&1&0\\2&1&3\end{vmatrix} + 1\begin{vmatrix}0&0&1\\-1&1&0\\2&1&3\end{vmatrix}\\
&= 2\left((-1)\begin{vmatrix}1&0&0\\1&0&0\\2&1&3\end{vmatrix} + 1\begin{vmatrix}1&0&0\\0&1&0\\2&1&3\end{vmatrix} + 0\begin{vmatrix}1&0&0\\0&0&1\\2&1&3\end{vmatrix}\right) + 0\\
&\quad +1\left((-1)\begin{vmatrix}0&0&1\\1&0&0\\2&1&3\end{vmatrix} + 1\begin{vmatrix}0&0&1\\0&1&0\\2&1&3\end{vmatrix}+0\begin{vmatrix}0&0&1\\0&0&1\\2&1&3\end{vmatrix}\right)\\
&= 0+(2)(1)\left(2\begin{vmatrix}1&0&0\\0&1&0\\1&0&0\end{vmatrix}+1\begin{vmatrix}1&0&0\\0&1&0\\0&1&0\end{vmatrix}+3\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}\right)+0\\
&\quad +(1)(-1)\left(2\begin{vmatrix}0&0&1\\1&0&0\\1&0&0\end{vmatrix}+1\begin{vmatrix}0&0&1\\1&0&0\\0&1&0\end{vmatrix}+3\begin{vmatrix}0&0&1\\1&0&0\\0&0&1\end{vmatrix}\right)\\
&\quad +(1)(1)\left(2\begin{vmatrix}0&0&1\\0&1&0\\1&0&0\end{vmatrix}+1\begin{vmatrix}0&0&1\\0&1&0\\0&1&0\end{vmatrix}+3\begin{vmatrix}0&0&1\\0&1&0\\0&0&1\end{vmatrix}\right)+0\\
&= 2(1)(3)(1)+(1)(-1)(1)(1)+1(1)(2)(-1) = 6-1-2=3.
\end{aligned}$$

2. A matrix is *upper triangular* if all of its entries below the diagonal are zero. What can you say about the determinant of a digaonal matrix?

Consider an upper triangular matrix $A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ 0 & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{nn} \end{bmatrix}$. Expanding along the first columns, we have

$$|A| = \alpha_{11}\begin{vmatrix} \alpha_{22} & \alpha_{23} & \dots & \alpha_{2n} \\ 0 & \alpha_{33} & \dots & \alpha_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{nn} \end{vmatrix} + 0 + \ldots + 0.$$

Then the surviving matrix is again upper triangular, and repeating n times we get

$$|A| = \alpha_{11}\alpha_{22}\ldots\alpha_{(n-1)(n-1)}|\alpha_{nn}|$$

where $|\alpha_{nn}|$ is the determinant of the 1 by 1 matrix $[\alpha_{nn}]$, which is just α_{nn}. Thus, we have proved:

Theorem: The determinant of an upper triangular matrix is the product of its entries along the diagonal.

3. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

using the adjoint A^*.

$$\begin{array}{lll} C_{11} = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0 & C_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 2 & C_{13} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \\ C_{21} = -\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 & C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 & C_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0 \\ C_{31} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1 & C_{32} = -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2 & C_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \end{array}$$

so we have

$$A^* = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 0 \\ -1 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

and

$$A^{-1} = \frac{1}{D(A)} A^* = \frac{1}{1\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 0 + 0} \begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix} = -1 \begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}.$$

Let's verify:

$$\begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0+1 & 0+0+0 & 0+0+0 \\ -2+0+2 & 0+1+0 & -2+2+0 \\ 1+0-1 & 0+0+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Given a complex number $z = a + ib$, consider the matrix $M_z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. For $z = a + ib$ and $w = x + iy$, compare M_{zw} and $M_z M_w$. Use what you've found to find a formula for $\frac{1}{z}$.

Since $zw = (a + ib)(x + iy) = ax + ibx + iay + i^2 by = (ax - by) + i(bx + ay)$, we have

$$M_{zw} = \begin{bmatrix} ax - by & -(bx + ay) \\ bx + ay & ax - by \end{bmatrix}.$$

On the other hand,

$$M_z M_w = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} ax - by & -ay - bx \\ bx + ay & -by + ax \end{bmatrix} = M_{zw}.$$

That is, multiplication of complex numbers works just like multiplication of matrices of these special forms. Then since we know how to find the inverse of a matrix, this gives us a formula for the inverse of a complex number: $A^{-1} = \frac{1}{D(A)} A^*$ says

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^* = \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \frac{1}{a^2 + b^2} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$$

and so

$$(a + ib)^{-1} = \frac{1}{a^2 + b^2}(a - ib).$$