1. Compute the determinant of the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ using row moves.

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -6 & -5 \\ 0 & 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -6 & -5 \\ 0 & 0 & -2 & -4 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -6 & -5 \\ 0 & 0 & 1 & 2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 7 \end{vmatrix}$$

$$= 14 \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 14 \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 14 \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 14 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 14 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 14 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 14.$$

2. Compute the determinant of the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ using a row expansion and then using a column expansion.

Expanding along the first row,

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{vmatrix} &= 1 \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} + 0 + 2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} + 0 \\ &= 1(-1) \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 0 + 0 + 2(1) \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 0 \\ &= -(1-2) + (-1-2) + 2(-1-2) = 1 - 3 - 6 = -8. \end{aligned}$$

Expanding along the 4th column:

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{vmatrix} &= 0 + 2 \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + 0 - \begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} \\ &= 2 \left[1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 0 + 2 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \right] - \left[1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 0 \right] \\ &= 2(0-2) - ((1+1) - (0-2)) = -4 - 4 = -8. \end{aligned}$$

3. Prove that the function $F_j(A) = \sum_{k=1}^{n} A_{jk}C_{jk}$ from lecture 22 is alternating.

We must show that switching the order of two rows in A reverses the sign of F_j. There are two cases: either row j is one of the rows being switched or not. If not, then we note that each C_{jk} is $(-1)^{j+k}M_{jk}$ and M_{jk} is a determinant, so switching two rows in A switches the same rows in M_{ij} and thus multiplies each term in F_j by (-1).

For the case when row j is switched with row i to obtain a new matrix A', let us illustrate the proof idea with an example. The idea is to use multilinearity of the determinant in the cofactor to expand in row i:

$$\begin{aligned} F_1\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= ae\begin{vmatrix} 1 & 0 \\ h & i \end{vmatrix} + af\begin{vmatrix} 0 & 1 \\ h & i \end{vmatrix} - bd\begin{vmatrix} 1 & 0 \\ g & i \end{vmatrix} - bf\begin{vmatrix} 0 & 1 \\ g & i \end{vmatrix} + cd\begin{vmatrix} 1 & 0 \\ g & h \end{vmatrix} + ce\begin{vmatrix} 0 & 1 \\ g & h \end{vmatrix} \end{aligned}$$

Now, switching rows 1 and 2,

$$\begin{aligned} F_1\left(\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}\right) &= d\begin{vmatrix} b & c \\ h & i \end{vmatrix} - e\begin{vmatrix} a & c \\ g & i \end{vmatrix} + f\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ &= db\begin{vmatrix} 1 & 0 \\ h & i \end{vmatrix} + dc\begin{vmatrix} 0 & 1 \\ h & i \end{vmatrix} - ea\begin{vmatrix} 1 & 0 \\ g & i \end{vmatrix} - ec\begin{vmatrix} 0 & 1 \\ g & i \end{vmatrix} + fa\begin{vmatrix} 1 & 0 \\ g & h \end{vmatrix} + fb\begin{vmatrix} 0 & 1 \\ g & h \end{vmatrix} \end{aligned}$$

The contribution to $F_j(A)$ with coefficient $A_{ji}A_{ij}$ and the contribution to $F_j(A')$ with coefficient $A_{ij}A_{ji}$ differ by a sign as well as by the column containing the 1 in the standard basis vector, but the resulting determinants are the same sincecould clear out the other entries in the column with row moves without changing the determinant.

4. State and prove a theorem about the determinant of orthogonal matrices.

If a matrix is orthogonal, we must have $AA^T = I$, so $D(AA^T) = D(I) = 1$. Then $D(A)D(A^T) = 1$. We also know that $D(A^T) = D(A)$, so if A is orthogonal we must have $D(A)^2 = 1$. Thus, we have proved

Theorem: $D(A) = \pm 1$ for matrices A with real entries.