Math 60 **Homework 6 Solutions**

1. Let $\langle (x_1,x_2),(y_1,y_2)\rangle = x_1x_2 + y_1y_2$. Is $\langle , \rangle$ an inner product? Explain.

It is not, since it is not bilinear:

$$\langle (\lambda x_1,\lambda x_2),(y_1,y_2)\rangle = (\lambda x_1)(\lambda x_2) + y_1y_2 = \lambda^2 x_1x_2 + y_1y_2$$

while

$$\lambda\langle (x_1,x_2),(y_1,y_2)\rangle. = \lambda x_1x_2 + \lambda y_1y_2.$$

2 Let $\langle (x_1,x_2),(y_1,y_2)\rangle = x_1y_1 - x_2y_2$. Is $\langle , \rangle$ an inner product? Explain.

This one is bilinear but not positive definite:

$$\langle (1,2),(1,2)\rangle = 1(1) - 2(2) = 1 - 4 = -3 < 0$$

and

$$\langle (1,1),(1,1)\rangle = (1)(1) - (1)(1) = 1 - 1 = 0 \quad \text{but} \quad (1,1) \neq (0,0).$$

Thus, not an innner product.

3. Using the inner product $\langle (x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)\rangle = 2x_1y_1 + 3x_2y_2 + x_3y_3 + x_4y_4$, find the distance and angle between the vectors $(1,1,0,1)$ and $(0,-1,-1,2)$ in $\mathbb{R}^4$.

$$\begin{aligned}
d((1,1,0,1),(0,-1,-1,2)) &= ||(1,1,0,1) - (0,-1,-1,2)|| \\
&= ||(1,2,1,-1)|| \\
&= \sqrt{\langle ((1,2,1,-1),(1,2,1,-1)\rangle)} \\
&= \sqrt{2(1)^2 + 3(2)^2 + 1^2 + (-1)^2} = \sqrt{2+12+1+1} = \sqrt{16} = 4
\end{aligned}$$

and the angle between the vectors is

$$\begin{aligned}
\theta((1,1,0,1),(0,-1,-1,2)) &= \cos^{-1}\left(\frac{\langle (1,1,0,1),(0,-1,-1,2)\rangle}{||(1,1,0,1)||\ ||(0,-1,-1,2)||}\right) \\
&= \cos^{-1}\left(\frac{2(1)(0) + 3(1)(-1) + 0(-1) + 1(2)\rangle}{\sqrt{2(1)^2 + 3(1)^2 + 0^2 + 1^2}\sqrt{2(0)^2 + 3(-1)^2 + (-1)^2 + 2^2}}\right) \\
&= \cos^{-1}\left(\frac{-1}{\sqrt{6}\sqrt{8}}\right) \\
&= \cos^{-1}\left(\frac{-1}{4\sqrt{3}}\right).
\end{aligned}$$

4. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an orthogonal matrix with respect to the standard basis. What can you say about a, b, c and d? (Hint: What does the fact that $A^TA = I$ say about a, b, c and d?)

If A is orthogonal with respect to the dot product, then we have $A^TA = I$:

$$A^TA = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so, we can say that for A to be orthogonal, we must have $a^2 + b^2 = 1$, $c^2 + d^2 = 1$ and $ac + bd = 0$. (For full credit, this is enough.)

We could additonally observe that:

- A^TA also must equal I, which gives us additonal equations $a^2+c^2=1$, $b^2+d^2=1$ and $ab+cd=0$.
- Then $a^2+c^2=a^2+b^2$ implies $c^2=b^2$ implies $c=b$ or $c=-b$.
- If $c=b$ then $ab+cd=ab+bd=b(a+d)=0$ and either $b=c=0$ or $a=-d$.
- If $c=-b$ then $ab+cd=ab-bd=b(a-d)=0$ and either $b=c=0$ or $a=d$.
- Moreover, $ab+cd=0$ says $ab=-cd$, so if a and b have the same sign, c and d must have opposite signs, and if a and b have opposite signs, c and d must have the same sign.
- Finally, $a^2+b^2=1$ says $b=\pm\sqrt{1-a^2}$ and $|a|\leq 1$, so 2 × 2 orthogonal matrices must have one of the forms

$$\begin{bmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{bmatrix}, \quad \begin{bmatrix} a & \sqrt{1-a^2} \\ -\sqrt{1-a^2} & a \end{bmatrix}, \quad \begin{bmatrix} a & -\sqrt{1-a^2} \\ \sqrt{1-a^2} & a \end{bmatrix},$$

$$\text{or} \quad \begin{bmatrix} a & -\sqrt{1-a^2} \\ -\sqrt{1-a^2} & -a \end{bmatrix}$$

where $-1\leq a\leq 1$.