1. Write the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ as a product of elementary matrices.

First, let us row-reduce:

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \xrightarrow{r_2 + r_1 \mapsto r_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1 - 2r_2 \mapsto r_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, we have an equation

$$I = E_{r_1 - 2r_2 \mapsto r_1} E_{r_2 + r_1 \mapsto r_2} A$$

which implies

$$E^{-1}_{r_2 + r_1 \mapsto r_2} E^{-1}_{r_1 - 2r_2 \mapsto r_1} = A$$

and since the inverse of an elementary matrix is the elementary matrix of the oppostive row operation, we have

$$A = E^{-1}_{r_2 + r_1 \mapsto r_2} E^{-1}_{r_1 - 2r_2 \mapsto r_1} = E_{r_2 - r_1 \mapsto r_2} E_{r_1 + 2r_2 \mapsto r_1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

We can verify:

$$\left[\begin{array}{cc} 1 & 0 \\ \hline -1 & 1 \end{array}\right] \left[\begin{array}{c|c} 1 & 2 \\ 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} 1(1) + 0(0) & 1(2) + 0(1) \\ -1(1) + 1(0) & -1(2) + 1(1) \end{array}\right] = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = A.$$

2. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ or show that A is not invertible.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & -1 \\ 0 & 0 & -5 & -2 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{5} & -\frac{2}{5} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{5} & -\frac{2}{5} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{5} & -\frac{2}{5} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{5} & -\frac{2}{5} \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$.

3. Use a change of basis matrix to write the vector $(1, -1, 2) \in \mathbb{R}^3$ as a linear combination of the vectors $\vec{u}_1 = (1, 1, 1), \vec{u}_2 = (0, 1, -1)$ and $\vec{u}_3 = (0, 0, 2)$.

Our vector $\vec{u}$ has standard-basis coordinates $(1, -1, 2)$; we must find the B-coordinates $\vec{u}_B$ where $B = ((1, 1, 1), (0, 1, -1), (0, 0, 2))$. Thus, we must find the change of basis matrix $M_B^S = (M_S^B)^{-1}$. Then since M_S^B is the matrix whose columns are B in standard coordinates, we have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right].$$

Thus $M_B^S = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and we have

$$\vec{u}_B = M_B^S \vec{u}_S = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ -1-1+0 \\ -1-\frac{1}{2}+1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -\frac{1}{2} \end{bmatrix}.$$

We can verify:

$$1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ 1-2+0 \\ 1+2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

4. Prove the *Shoe-Sock Theorem*: If A and B are invertible matrices then their product AB is invertible with $(AB)^{-1} = B^{-1}A^{-1}$.

Here we have only to check that $(AB)(B^{-1}A^{-1}) = I$. But

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = (AI)A^{-1} = AA^{-1} = I$$

as required.