1. Find a basis for the kernel of

$$A = \begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 2 & 2 & 1 & 1 & -1 \\ 2 & 0 & 1 & 1 & -1 \\ 1 & 2 & -1 & 1 & 1 \end{bmatrix} \in M_{4,5}(\mathbb{R})$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 2 & 2 & 1 & 1 & -1 \\ 2 & 0 & 1 & 1 & -1 \\ 1 & 2 & -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 0 & -2 & -3 & 1 & 3 \\ 2 & 0 & 1 & 1 & -1 \\ 1 & 2 & -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 0 & -2 & -3 & 1 & 3 \\ 0 & -4 & -3 & 1 & 3 \\ 1 & 2 & -1 & 1 & 1 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 0 & -2 & -3 & 1 & 3 \\ 0 & -4 & -3 & 1 & 3 \\ 0 & 0 & -3 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & -4 & -3 & 1 & 3 \\ 0 & 0 & -3 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 3 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 2 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we have free variables $x_4 = \alpha$ and $x_5 = \beta$, with $x_1 = -\frac{2}{3}\alpha$, $x_2 = 0$ and $x_3 = \frac{1}{3}\alpha + \beta$, so solution vectors have the form

$$\left(-\frac{2}{3}\alpha, 0, \frac{1}{3}\alpha + \beta, \alpha, \beta\right) = \alpha\left(-\frac{2}{3}, 0, \frac{1}{3}, 1, 0\right) + \beta(0, 0, 1, 0, 1)$$

and the kernel has basis

$$\left\{\left(-\frac{2}{3}, 0, \frac{1}{3}, 1, 0\right), (0, 0, 1, 0, 1)\right\}.$$

2. Find all solutions to the system of linear equations in $\mathbb{R}$:

$$\begin{array}{rrrrcr} x & +2y & -z & +w & = & 1 \\ 2x & +y & +z & +w & = & 2 \\ -x & & -2z & -w & = & -1 \end{array}$$

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 \\ -1 & 0 & -2 & -1 & -1 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & -3 & 3 & -1 & 0 \\ -1 & 0 & -2 & -1 & -1 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & -3 & 3 & -1 & 0 \\ 0 & 2 & -3 & 0 & 0 \end{array}\right] \to$$

$$\to \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 2 & -3 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -3 & -2 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & 0 \end{array}\right] \to$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 0 & \frac{5}{3} & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{3} & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \end{array}\right] \to$$

The homogeneous system has one free variable $x_4 = \alpha$ with solution vectors $\left(\frac{1}{3}\alpha, -\alpha, -\frac{2}{3}\alpha, \alpha\right)$; the particular solution is $\vec{u} = (1,0,0,0)$, so the set of all solutions has the form

$$(1,0,0,0) + \text{Span}\left(\left(\frac{1}{3}, -1, -\frac{2}{3}, 1\right)\right).$$

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x,y,z) = (2x+y, x+y+z, 2x+y+z)$. Is f a linear transformation? Explain.

We must verify that $f(\vec{u}+\vec{v}) = f(\vec{u}) + f(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^3$ and that $f(\alpha\vec{u}) = \alpha f(\vec{u})$ for all $\vec{u} \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Let $\vec{u} = (x,y,z)$ and $\vec{v} = (x', y', z')$. Then:

$$\begin{aligned} f(\vec{u}+\vec{v}) &= f((x,y,z) + (x',y',z')) \\ &= f((x+x', y+y', z+z')) \\ &= (2(x+x') + (y+y'), (x+x') + (y+y') + (z+z'), 2(x+x') + (y+y') + (z+z')) \\ &= (2x+2x'+y+y', x+x'+y+y'+z+z', 2x+2x'+y+y'+z+z') \\ &= (2x+y, x+y+z, 2x+y+z) + (2x'+y', x'+y'+z', 2x'+y'+z') \\ &= f(\vec{u}) + f(\vec{v}) \end{aligned}$$

and

$$\begin{aligned} f(\alpha\vec{u}) &= f(\alpha(x,y,z)) \\ &= f(\alpha x, \alpha y, \alpha z) \\ &= (2\alpha x + \alpha y, \alpha x + \alpha y + \alpha z, 2\alpha x + \alpha y + \alpha z) \\ &= \alpha(2x+y, x+y+z, 2x+y+z) \\ &= \alpha f(\vec{u}) \end{aligned}$$

so f is indeed a linear transformation.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x,y) = (x-y, x+y, -2x+y)$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $g(u,v,w) = (u-v+w, 2u+v-w)$. Find the matrices of f, g, $f \circ g$. and $g \circ f$.

The matrices of f and g respectively are

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix};$$

then the matrices of $f \circ g$ and $g \circ f$ respectively are

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)-1(2) & 1(-1)-1(1) & 1(1)-1(-1) \\ 1(1)+1(2) & 1(-1)+1(1) & 1(1)+1(-1) \\ -2(1)+1(2) & -2(-1)+1(1) & -2(1)+1(-1) \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 & 2 \\ 3 & 0 & 0 \\ 0 & 3 & -3 \end{bmatrix} \end{aligned}$$

and

$$
\begin{aligned}
BA &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1(1) - 1(1) + 1(-2) & 1(-1) - 1(1) + 1(1) \\ 2(1) + 1(1) - 1(-2) & 2(-1) + 1(1) - 1(1) \end{bmatrix} \\
&= \begin{bmatrix} -2 & -1 \\ 5 & -2 \end{bmatrix}
\end{aligned}
$$