1. Find a basis for the span of $\{(1,2,3,1),(2,0,1,-1),(3,2,4,0),(1,-2,-2,-2)\} \subset \mathbb{R}^4$.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & -1 \\ 3 & 2 & 4 & 0 \\ 1 & -2 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -4 & -5 & -3 \\ 3 & 2 & 4 & 0 \\ 1 & -2 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -4 & -5 & -3 \\ 0 & -4 & -5 & -3 \\ 1 & -2 & -2 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -4 & -5 & -3 \\ 0 & -4 & -5 & -3 \\ 0 & -4 & -5 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -4 & -5 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & -5 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -4 & -5 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\mathrm{Span}(\{(1,2,3,1),(2,0,1,-1),(3,2,4,0),(1,-2,-2,-2)\})$ has basis $\{(1,2,3,1),(0,-4,-5,-3)\}$.

2. Find a basis for the span of $\{(1,2,4,2),(2,0,1,4),(3,1,4,1),(2,1,1,2)\} \subset (\mathbb{Z}_5)^4$.

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & 0 & 1 & 4 \\ 3 & 1 & 4 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 1 & 3 & 3 \\ 3 & 1 & 4 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

so $\mathrm{Span}((1,2,4,2),(2,0,1,4),(3,1,4,1),(2,1,1,2)) \subset (\mathbb{Z}_5)^4$ has basis $\{(1,2,4,2),(0,1,3,3),(0,0,2,0),(0,0,0,2)\}$; also, since the span has dimension 4, the original spanning set is also a basis.

3. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 3 & 1 & -2 & 1 & 3 \\ -1 & 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix} \in M_{4,5}(\mathbb{R})$.

$$\begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 3 & 1 & -2 & 1 & 3 \\ -1 & 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & -5 & -5 & 1 & 6 \\ -1 & 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & -5 & -5 & 1 & 6 \\ 0 & 3 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 3 & 3 & 1 & 2 \\ 0 & -5 & -5 & 1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & -5 & -1 \\ 0 & -5 & -5 & 1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & 11 & 11 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Since the rank of a matrix is the dimension of its row space, this matrix has rank 4.

4. Find the dimension of $S \cap T \subset \mathbb{R}^4$ where

$$S = \text{Span}((1,1,2,1),(0,1,-1,1)) \subset \mathbb{R}^4 \quad \text{and} \quad T = \text{Span}((-1,1,1,0),(0,3,2,2)) \subset \mathbb{R}^4.$$

<u>$\dim(S)$:</u>

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \quad \text{is in echelon form} \quad \Rightarrow \dim(S) = 2.$$

<u>$\dim(T)$:</u>

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 2 \end{bmatrix} \quad \text{is in echelon form} \quad \Rightarrow \dim(T) = 2.$$

<u>$\dim(S+T)$:</u>

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 5 & -1 \\ 0 & 3 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 5 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\dim(S+T) = 3$ and we have $\dim(S \cap T) = \dim(S) + \dim(T) - \dim(S+T) = 2 + 2 - 3 = 1$. Thus S and T are two planes in $\mathbb{R}^4$ which intersect in a line.