1. Let $S = \{(\alpha, \beta, \alpha + 1) \mid \alpha, \beta \in \mathbb{R}\}$. Is S a subspace? Explain.

S is *not* a subspace since it is not closed under either vector addition

$$(\alpha, \beta, \alpha 1) + (\alpha', \beta', \alpha' + 1) = (\alpha + \alpha', \beta + \beta', \alpha + \alpha' + 2) \notin S$$

or scalar multiplication:

$$\lambda(\alpha, \beta, \alpha + 1) = (\lambda\alpha, \lambda\beta, \lambda\alpha + \lambda) \notin S \text{ if } \lambda \neq 1.$$

2. Find all subspaces of $(\mathbb{Z}_3)^2$.

Note that $(\mathbb{Z}_3)^2 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$ so we could in principle check by brute force, simply listing all subsets and testing each one for closure under vector addition and under scalar multiplication. However, if we think in terms of spans, we will find the problem much simpler. For example, every subspace containing $(1,0)$ must also contain $2(1,0) = (2,0)$. Let us list the subspaces spanned by the single vectors:

- $\text{Span}((0,0)) = \{(0,0)\}$
- $\text{Span}((1,0)) = \{(0,0),(1,0),(2,0)\} = \text{Span}((2,0))$
- $\text{Span}((0,1)) = \{(0,0),(0,1),(0,2)\} = \text{Span}((0,2))$
- $\text{Span}((1,1)) = \{(0,0),(1,1),(2,2)\} = \text{Span}((2,2))$
- $\text{Span}((1,2)) = \{(0,0),(1,2),(2,1)\} = \text{Span}((2,1))$

Now let us consider spans of pair of nonzero vectors. If the two vectors in our pair are scalar multiples of each other, then they span one of the spaces listed above; on the other hand, any pair of linearly independent vectors in the space will span the whole space $(\mathbb{Z}_3)^2$: there are six pairs to check, namely

$$((1,0),(0,1)),((1,0),(1,1)),((1,0),(1,2)),((0,1),(1,1)),((0,1),(1,2)) \quad \text{and} \quad ((1,1),(1,2)).$$

Taking for example the case of $\text{Span}((1,0),(0,1))$, the span contains the scalar mutiples of $(1,0)$ and $(0,1)$, i.e., $\{(0,0),(1,0),(2,0),(0,1),(0,2)\}$, as well as sums of these: $(1,1) = (1,0)+(0,1)$, $(1,2) = (1,0)+(0,2)$, $(2,1) = (2,0)+(0,1)$, and $(2,1) = (2,0)+(0,2)$, so $\text{Span}((1,0),(0,1)) = (\mathbb{Z}_3)^2$. The other cases are similar.

Finally, we don't need to consider span of triples or larger sets of vectors since these each contain a pair. Hence, there are six subspaces in total.

3. Identify a spanning set for the subspace $S = \{(\alpha, \beta, \alpha - 2\beta, \beta, 3\alpha)\}$ of $\mathbb{R}^5$.

$$\begin{aligned}(\alpha, \beta, \alpha - 2\beta, \beta, 3\alpha) &= (\alpha, 0, \alpha, 0, 3\alpha) + (0, \beta, -2\beta, \beta, 0)\\ &= \alpha(1,0,1,0,3) + \beta(0,1,-2,1,0)\end{aligned}$$

so $S = \text{Span}((1,0,1,0,3),(0,1,-2,1,0))$.

4. Is the set $\{(1,3,2),(2,1,1),(1,0,1),(4,2,3)\} \subset \mathbb{R}^3$ linearly independent or linearly dependent? Explain.

$\mathbb{R}^3$ is spanned by the three vectors $\{(1,0,0),(0,1,0),(0,0,1)\}$; since our set has four vectors and $4 > 3$, the set is linearly dependent.