

1. An element in a number system is called a *unit* if it has a multiplicative inverse. Identify the units of \mathbb{Z}_9 by finding their multiplicative inverses. For each non-unit element, show that no element of \mathbb{Z}_9 works as a multiplicative inverse.

α	α^{-1}	Justification
1	1	$1(1) = 1$
2	5	$2(5) = 10 = 1$
3	--	$3(1) = 3 \neq 1, (2) = 6 \neq 1, 3(4) = 12 = 3 \neq 1, 3(5) = 15 = 6 \neq 1,$ $3(6) = 18 = 0 \neq 1, 3(7) = 21 = 3 \neq 1, 3(8) = 24 = 6 \neq 1$
4	7	$4(7) = 28 = 1$
5	2	$5(2) = 10 = 1$
6	--	$6(1) = 6 \neq 1, 6(2) = 12 = 3 \neq 1, 6(3) = 18 = 0 \neq 1, 6(4) = 24 = 6 \neq 1,$ $6(5) = 30 = 3 \neq 1, 6(6) = 36 = 0 \neq 1, 6(7) = 42 = 6 \neq 1, 6(8) = 48 = 3 \neq 1$
7	4	$7(4) = 28 = 1$
8	8	$8(8) = 64 = 1$

2. A nonzero element of a number system $\beta \neq 0$ is a *zero divisor* if there is a nonzero element $\alpha \neq 0$ such that $\alpha\beta = 0$. Prove that if \mathbb{F} is a field, then \mathbb{F} contains no zero divisors.

Proof: Let \mathbb{F} be a field and suppose $\alpha\beta = 0$ and $\alpha \neq 0$. Then there is an $\alpha^{-1} \in \mathbb{F}$ such that $\alpha\alpha^{-1} = 1$; then $\alpha\beta = 0$ implies

$$\beta = 1\beta = (\alpha^{-1}\alpha)\beta = \alpha^{-1}(\alpha\beta) = \alpha^{-1}0 = 0.$$

Thus, for any $\alpha \neq 0 \in \mathbb{F}$, $\alpha\beta = 0$ implies $\beta = 0$ and α is not a zero divisor. \square

Note that one can also approach this as a proof by contradiction in a few different ways.

3. Let \mathbb{F} be a field and $\alpha, \beta, \gamma \in \mathbb{F}$. Using the field axioms, show that if $\alpha\beta = \alpha\gamma$ and $\alpha \neq 0$, then $\beta = \gamma$.

Proof: Since \mathbb{F} is a field and $\alpha \neq 0$, there is an $\alpha^{-1} \in \mathbb{F}$ such that $\alpha^{-1}\alpha = 1$. Then since $\alpha\beta = \alpha\gamma$, we have

$$\alpha^{-1}(\alpha\beta) = \alpha^{-1}(\alpha\gamma)$$

which then says

$$(\alpha^{-1}\alpha)\beta = (\alpha^{-1}\alpha)\gamma$$

since multiplication in \mathbb{F} is commutative; but then we have $1\beta = 1\gamma$, i.e., $\beta = \gamma$, as required. \square

4. Prove by induction that $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$.

Proof:

Base case: Let $n = 1$. Then our statement says

$$\sum_{i=1}^1 i^3 = 1^3 = 1 = \frac{4}{4} = \frac{1^2(1+1)^2}{4}.$$

Induction step: Let us suppose that $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$ and consider $\sum_{i=1}^{n+1} i^3$; we must show that this is equal to $\frac{(n+1)^2(n+2)^2}{4}$. Well,

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^n i^3 + (n+1)^3$$

$$\begin{aligned} &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} \\ &= \frac{n^2(n+1)^2 + (4n+4)(n+1)^2}{4} \\ &= \frac{(n^2+4n+4)(n+1)^2}{4} \\ &= \frac{(n+2)^2(n+1)^2}{4}, \end{aligned}$$

as required.

□