1. Let B be the bilinear form $B((u_1,u_2),(v_1,v_2)) = 2u_1v_1 - u_2v_1 + 3u_1v_2 - u_2v_2$. Find the matrix A such that $B(\vec{u},\vec{v}) = \vec{u}^T A\vec{v}$ where $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Let us write $A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$. Then we have

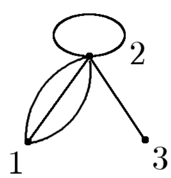
$$B(\vec{u},\vec{v}) = \vec{u}^T A\vec{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha_{11}u_1v_1 + \alpha_{12}u_2v_1 + \alpha_{21}u_1v_2 + \alpha_{22}u_2v_2$$

so we have $A = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$.

2. Find $\vec{u}^*(\vec{v})$ if $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ using the bilinear form in problem 1.

$$B(\vec{u},\vec{v}) = 2(3)(1) - (1)(1) + 3(3)(-2) - (1)(-2) = 6 - 1 - 18 + 2 = -11.$$

3. How many paths of length 4 from vertex 2 to vertex 3 are there in the graph below?



The adjacency matrix is $A = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ so we have

$$A^4 = \begin{bmatrix} 99 & 63 & 33 \\ 63 & 131 & 21 \\ 33 & 21 & 11 \end{bmatrix}$$

and there are 21 paths of length 4 from vertex 2 to vertex 3.

4. Find a stable state for the stochastic matrix $A = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.4 & 0.4 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}$.

We need an eignevector for A^T with $\lambda = 1$. Then row-reducing, we have

$$\begin{bmatrix} 0.5-1 & 0.3 & 0.2 \\ 0.3 & 0.4-1 & 0.3 \\ 0.2 & 0.4 & 0.4-1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the eigenspace is Span((1, 1, 1)) and the vector (1, 1, 1) is a stable state distribution.