1. True or False: The set $\{(1,2,-1),(1,0,2),(1,1,1),(1,1,0)\} \subset \mathbb{R}^3$ is linearly independent. Explain.

False. This is a set of four vectors in a three-dimensional space, and we have a theorem which says that every set of more than n vectors in an n-dimensional space is linearly dependent.

2. Find the dimension of $S \cap T$ if $S = \{(\alpha, \alpha+\beta, 2\alpha) \mid \alpha, \beta \in \mathbb{R}\}$ and $T = \{(\beta, 2\beta, \alpha) \mid \alpha, \beta \in \mathbb{R}\}$.

We have $S = \mathrm{Span}((1,1,2),(0,1,0))$ and $T = \mathrm{Span}((1,2,0),(0,0,1))$, so we have

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\dim(S+T) = 3$, so $\dim(S \cap T) = \dim(S) + \dim(T) - \dim(S+T) = 2+2-3 = 1$.

3. True or False: The set $\{(\alpha, \beta+\alpha, \alpha-\beta, \alpha+2) \mid \alpha, \beta \in \mathbb{R}\} \subset \mathbb{R}^4$ is a subspace. Explain.

This is **not** a subspace since subspaces must be closed under scalar multiplication, but for example $\vec{u} = (1, 1+1, 1-1, 1+2) = (1,2,0,3) \in S$ but $2\vec{u} = (2,4,0,6) \notin S$ since then we'd need to have $\alpha = 2$ and $\beta = 2$ to satisfy the first three components, but then the fourth component would have to be $\alpha + 2 = 2+2 = 4 \neq 6$.

4. Find a basis for the kernel of the linear transformation $f : \mathbb{R}^6 \to \mathbb{R}^3$ defined by

$$f(x,y,z,u,v,w) = (x+y+z+2u-v, y+3z-u+v+w, 2x+y+u-v-w).$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 1 \\ 2 & 1 & 0 & 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 1 \\ 0 & -1 & -2 & -3 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 1 \\ 0 & 0 & 1 & -4 & 2 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 11 & -5 & 1 \\ 0 & 0 & 1 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 6 & -3 & 0 \\ 0 & 1 & 0 & 11 & -5 & 1 \\ 0 & 0 & 1 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -5 & 2 & -1 \\ 0 & 1 & 0 & 11 & -5 & 1 \\ 0 & 0 & 1 & -4 & 2 & 0 \end{bmatrix}$$

so setting $x_4 = \alpha$, $x_5 = \beta$ and $x_6 = \gamma$, we have $x_1 = 5\alpha - 2\beta + \gamma$, $x_2 = -11\alpha + 5\beta - \gamma$, and $x_3 = 4\alpha - 2\beta$; thus solution vectors have the form

$$(5\alpha - 2\beta + \gamma, -11\alpha + 5\beta - \gamma, 4\alpha - 2\beta, \alpha, \beta, \gamma) = \alpha(5,-11,4,1,0,0) + \beta(-2,5,2,0,1,0) + \gamma(1,-1,0,0,0,1)$$

and the kernel has basis $\{(5,-11,4,1,0,0), (-2,5,2,0,1,0), (1,-1,0,0,0,1)\}$.

5. Give an example of a bilinear form on $\mathbb{R}^2$ which is not an inner product.

An inner product is a positive definite bilinear form, i.e. a bilinear form such that $\langle \vec{u}, \vec{u} \rangle \geq 0$ with equality iff $\vec{u} = \vec{0}$. Then for example $\langle (u_1,u_2),(v_1,v_2) \rangle = u_1 v_1$ is a bilinear form which is not positive definite since $\langle (0,1),(0,1) \rangle = 0(0) = 0$ but $(0,1) \neq (0,0)$.

Note that many examples are possible.

6. Compute the determinant of $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix} \in M_4(\mathbb{Z}_3)$.

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{vmatrix} = -1\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix} = (-1)(1)\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 2(1)\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} - 2(2)\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$$
$$= -1(2-1) - 2(0) - 4(4-2) = -1 - 0 - 8 = -9 = 0.$$

7. Recall that a matrix A is *nilpotent* if $A^n = 0$ for some $n > 1$. What can you say about the determinant of a nilpotent matrix? Explain.

If A is nilpotent, then we have $D(A^n) = D(A)^n = D(0) = 0$, so $D(A)$ is an nth root of zero. Since $D(A)$ is an element of a field, $D(A)^n = 0$ can only happen if $D(A) = 0$, so we must have $D(A) = 0$.

8. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 2 & 2 & 1 \end{bmatrix} \in M_3(\mathbb{Z}_5)$.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 3 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 2 & 0 \\ 0 & 0 & 3 & 2 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 2 & 0 \\ 0 & 0 & 1 & 4 & 1 & 2 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 4 & 0 \\ 0 & 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & 1 & 2 \end{array}\right]$$

so we have $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 2 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 3 & 2 \\ 4 & 1 & 2 \end{bmatrix}$

9. Find the eigenvalues and eigenspaces of the matrix $A = \begin{bmatrix} 1 & -4 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in M_3(\mathbb{R})$.

Computing the characteristic polynomial, we have

$$\chi_A = \begin{vmatrix} 1-\lambda & -4 & -1 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)^2$$

so the eigenvalues are $\lambda = 1$ and $\lambda = -1$.

<u>$\lambda = 1$:</u>

$$\begin{bmatrix} 1-1 & -4 & -1 \\ 0 & -1-1 & 0 \\ 0 & 0 & -1-1 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the $\lambda = 1$ eigenspace is $\text{Span}((1,0,0))$.

<u>$\lambda = -1$:</u>

$$\begin{bmatrix} 1+1 & -4 & -1 \\ 0 & -1+1 & 0 \\ 0 & 0 & -1+1 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the $\lambda = -1$ eigenspace is $\text{Span}((2,1,0),(\frac{1}{2},0,1))$.

10. A linear transformation f has elementary divisors $(x^2+1)^2$ and $(x+1)^3$. Write the rational canonical form for the matrix of f.

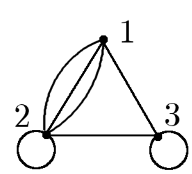
$$A = \left[\begin{array}{cccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right].$$

11. Find the matrix $(A \oplus B) \otimes C$ if $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$.

$$A \oplus B = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

so

$$(A \oplus B) \otimes C = \begin{bmatrix} 3 & 2 & 6 & 4 & 3 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 4 & 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 2 & -3 & -2 & 3 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & -2 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -1 & -2 \end{bmatrix}.$$

12. How many paths of length 4 from vertex 1 to vertex 3 are there in the graph  ?

The graph has adjacency matrix $A = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ so we have

$$A^2 = \begin{bmatrix} 10 & 4 & 4 \\ 4 & 11 & 5 \\ 4 & 5 & 3 \end{bmatrix}, A^3 = \begin{bmatrix} 16 & 38 & 18 \\ 38 & 28 & 20 \\ 18 & 20 & 12 \end{bmatrix} \quad \text{and} \quad A^4 = \begin{bmatrix} 132 & 104 & 72 \\ 104 & 162 & 86 \\ 72 & 86 & 50 \end{bmatrix}$$

so there are 72 paths of length 4 from vertex 1 to vertex 3.

Bonus [0-2 points each]:

A. Find all solutions to the system of equations $\begin{array}{ccccccc} x & +y & -z & +w & = & 1 \\ 2x & -y & & +w & = & 2 \\ x & -y & +z & & = & -1 \end{array}$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & -1 & 0 & 1 & 2 \\ 1 & -1 & 1 & 0 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & -3 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & -3 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 1 & 0 \\ 0 & -2 & 2 & -1 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & -2 & 2 & -1 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 2 & -1 & -6 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 2 & -1 & -6 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} & -3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} & -3 \end{array}\right]$$

Thus the set of solutions is the affine subspace $(0,-2,-3,0) + \mathrm{Span}(-\frac{1}{2}, 0, \frac{1}{2}, 1)$.

B. Find the companion matrix for the polynomial $x^4 + 2x^2 + 1$.

First, we must factor the polynomial as a product of powers of irreducible polynomials. We have $x^4 + 2x^2 + 1 = (x^2+1)^2$ with companion matrix $A = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ if our field is $\mathbb{Q}$ or $\mathbb{R}$; if our field is $\mathbb{C}$ then we have $x^4 + 2x^2 + 1 = (x+i)^2(x-i)^2$ and the companion matrix is $A = \begin{bmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{bmatrix}$.

C. Find the minimal polynomial of $\vec{v} = (1,2)$ with respect to the matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$.

We have $A\vec{v} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$ and $A^2\vec{v} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 19 \\ -7 \end{bmatrix}$. Then we have

$$\begin{bmatrix} 1 & 8 & 19 \\ 2 & 1 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 8 & 19 \\ 0 & -15 & -45 \end{bmatrix} \to \begin{bmatrix} 1 & 8 & 19 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\mu_A = 5 - 3x + x^2.$$